

Local cloning of genuine entangled states of three qubit

Sujit K. Choudhary,^{1,*} Guruprasad Kar,^{1,†} Samir Kunkri,^{2,‡} Ramij Rahaman,^{1,§} and Anirban Roy^{3,¶}

¹*Physics and Applied Mathematics Unit, Indian statistical Institute, 203, B. T. Road, Kolkata 700 108, India*

²*Mahadevananda Mahavidyalaya, Monirampore, Barrackpore, North 24 Parganas, 700120*

³*17, Bhupen Bose Avenue, Kolkata 700004, India*

We discuss (im)possibility of the exact cloning of orthogonal but genuinely entangled three qubit states aided with entangled ancilla under local operation and classical communication. Whereas any two orthogonal GHZ states taken from the canonical GHZ basis, can be cloned with the help of a known GHZ state, surprisingly we find that no two W states can be cloned by using any known three qubit (possibly entangled) state as blank copy.

PACS numbers:

Manipulation of multiparty entanglement by local operation and classical communication (LOCC) is an open area in quantum information. Distinguishability of orthogonal entangled states by LOCC and their cloning under LOCC with the help of appropriate entanglement are two important and closely connected areas in this field. Some interesting results have been obtained in case of LOCC discrimination of orthogonal entangled states [1, 2, 3, 4]. The concept of entanglement cloning under LOCC aided with entanglement, henceforth will be called as **local cloning**, is a newly emerging area which was first introduced by Ghosh *et. al.* [5]. Since then many works have been done in this direction [6, 7, 8]. These sort of works are important not only due to the fact that these are helpful in understanding the nonlocality of a set [9] but also of the fact that local cloning is very closely connected with many important information processing tasks, like channel copying, entanglement distillation, error correction and quantum key distribution [7]. But most of these works deal mainly with maximally entangled states of two qubits. Recently, Choudhary *et. al.* [10] have discussed the impossibility of local cloning of arbitrary entangled states shared between two parties. They, by entanglement considerations have obtained the necessary amount of entanglement in blank copy for exact local cloning of two orthogonal nonmaximally entangled bipartite states. This work has given rise the potency to explore the possibility of local cloning of multipartite entangled states which in fact is of greater interest. Although some results are known for local discrimination of a set of orthogonal multiparty entangled state, no result is known for local copying. We, in this letter concentrate on three qubit pure states. W and GHZ states are the two extreme representatives of the inequivalent kinds of genuine three qubit entangled states [11]. Our result shows that whereas any two GHZ states from the canonical set of eight orthogonal GHZ states can be cloned locally with the help of a GHZ state as the blank copy, no two W-states, taken from the complete set of orthogonal W-states, can be cloned with the help of any 3-qubit entangled state. We also find the condition under which a set of 3 orthogonal GHZ states cannot be cloned with

the help of any 3-qubit entangled state.

The full orthogonal canonical set of tripartite GHZ states can be written as (upto a global phase):

$$|\Psi_{p,i,j}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|0\ i\ j\rangle + (-1)^p |1\ \bar{i}\ \bar{j}\rangle], \quad (1)$$

where $p, i, j = 0, 1$ and a bar over a bit value indicates its logical negation.

Consider any pair from (1). Let one state of this pair is shared among three parties; Alice, Bob and Charlie. They share another known GHZ state as blank copy. It can be easily shown that control NOT (CNOT) operation ($C|i\rangle|j\rangle = |i\rangle|(j+i) \bmod 2\rangle$) by each of the parties will make cloning possible.

Existence of the three GHZ states that cannot be cloned by LOCC:

We would like to mention a necessary condition for cloning of a 3 qubit entangled state under LOCC with the help of a 3 qubit state as blank copy which is required for our investigation.

A necessary condition for cloning of a 3 qubit state under the usual LOCC (where all the three qubits of the state is operated separately) would be: “The states should remain copiable when the two qubits are operated jointly at one place whereas the third undergo a separate local operation at a different place and there can be classical communication between these two places”.

Consider three states from the set (1). The first two qubits of these states are put together in lab-A whereas the remaining third qubit in a different lab (lab-B).

These three states are equivalent to the three Bell states in the above mentioned bipartite cut if and only if, all of them have same i and two among them have same j but different p . As three Bell states cannot be cloned by an amount of entanglement less than $\log_2 3$ ebit [10] and as 1 ebit is the maximum bipartite entanglement that a 3-qubit state can have, so we conclude that these GHZ states with any 3 qubit ancilla state cannot be cloned by LOCC. Any set of three states which are not in the above form in any bipartite cut can always be cloned by LOCC using a known GHZ state as ancilla, where every party uses CNOT.

Our main objective in this letter is to explore the possibility of cloning of W-states under LOCC.

A full set of tripartite $|W\rangle$ states is given as

$$|W_1\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle + |100\rangle + |111\rangle)$$

$$|W_2\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle + |101\rangle + |110\rangle)$$

$$|W_3\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle - |100\rangle + |010\rangle)$$

$$|W_4\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle - |101\rangle + |000\rangle)$$

$$|W_5\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle - |010\rangle - |111\rangle)$$

$$|W_6\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle - |000\rangle - |110\rangle)$$

$$|W_7\rangle_{123} = \sqrt{\frac{1}{3}}(|100\rangle - |111\rangle + |010\rangle)$$

$$|W_8\rangle_{123} = \sqrt{\frac{1}{3}}(|101\rangle - |110\rangle + |000\rangle)$$

We put our main result in the following theorem.

Theorem : No set of orthogonal W-states can be cloned by LOCC with the help of any three qubit state as blank copy.

Lemma : States belonging to W-class, unless it is a W state, has at least one bipartite cut for which the entanglement, the ‘Bipartite Entanglement’, $E < -\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3}$.

Proof : A generic W-class state[11] shared among three parties is:

$$|\Psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle$$

where $a, b, c > 0$, and $d \equiv 1 - (a + b + c) \geq 0$.

If possible, let in all the three bipartite cuts, the entanglement of the above W-class state is greater than or equal to $\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3}$.

The entanglement in 1 vs 2-3 cut $E_{1:23}$ is :

$$-\frac{1 - \sqrt{(1-2c)^2 + 4cd}}{2} \log_2 \frac{1 - \sqrt{(1-2c)^2 + 4cd}}{2}$$

$$-\frac{1 + \sqrt{(1-2c)^2 + 4cd}}{2} \log_2 \frac{1 + \sqrt{(1-2c)^2 + 4cd}}{2}$$

$$\text{Now } E_{1:23} \geq -\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3} \implies,$$

$$\frac{1}{3} \leq \frac{1 \pm \sqrt{(1-2c)^2 + 4cd}}{2} \leq \frac{2}{3}; \quad \text{i.e. } \frac{1}{3} \leq c \leq \frac{2}{3} \quad (2)$$

Similarly, for other cuts, the previous assumption will lead to

$$\frac{1}{3} \leq b \leq \frac{2}{3} \quad \text{and} \quad \frac{1}{3} \leq a \leq \frac{2}{3} \quad (3)$$

Both the inequalities (2) and (3), can not hold simultaneously, unless $d=0$ and $a=b=c$ (i.e. a W-state). QED.

Proof of the Theorem : One needs an entangled blank state to clone an entangled state or else, entanglement of the entire system will increase under LOCC which is impossible. So, let us try to clone the W-states with the help of a known genuine tripartite entangled state as the blank copy. Recently, Dür et. al. [11] have shown that any genuine tripartite entangled state can have entanglement either of the W-kind or of GHZ-kind. So our blank copy is either of W-class or of GHZ-class.

(i) **Blank copy having GHZ-kind of entanglement.**

In this case we will show that even a known W state cannot be cloned by LOCC. The minimum number of product terms for a given state cannot be altered by LOCC [11]. But such a cloning would imply that the minimal no. of product term for a given state is increased from 6 (minimum no. of product term in the input to the cloner) to 9 (corresponding no. in the output), by LOCC which is impossible.

(ii) **Blank copy having W-kind of entanglement.**

We first consider a W-class state which is not a W-state as our blank copy. Here too a known W state cannot be cloned by LOCC. Keeping the lemma in mind, we consider a situation where those two qubits of the blank copy are kept together in lab-A for which the entanglement in that bi-partite cut of the blank copy (W-kind of state in this case) is less than that of corresponding W-state (the state proposed to be cloned). Corresponding qubits of the state to be cloned are also put in Lab-A. Another lab (lab-B) contains the remaining third qubits of these states. As LOCC cannot increase entanglement hence the W-state is not copiable under LOCC between these labs. But as mentioned earlier this is necessary for local cloning of any 3-qubit state, hence we conclude that a W-kind of state (unless it is a W-state) is not helpful in LOCC-cloning of W-states.

(iii) **W-state as Blank copy**

In this case we will prove the theorem by showing the impossibility of cloning any pair of W states from the above mentioned W-basis by LOCC. There are 28 such

pairs. Consider one pair- $|W_m\rangle_{123}$ and $|W_n\rangle_{123}$. Let E_{ij}^{mn} denotes the subspace generated by the support of ρ_{ij}^m and ρ_{ij}^n , where $\rho_{ij}^m = \text{Tr}_k\{|W_m\rangle_{123}\langle W_m|\}$ (similar is ρ_{ij}^n). Here $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. A close inspection will reveal that these pairs fall broadly into three categories:

(A) pairs for which $\dim(E_{ij}^{mn}) = 2$ for at least one value of k are :

(a) $(W_1, W_2), (W_3, W_6)$ for $k = 2$, (b) $(W_1, W_4), (W_6, W_7)$ for $k = 1$, (c) $(W_2, W_7), (W_3, W_4)$ for $k = 3$.

(B) pairs for which $\dim(E_{ij}^{mn}) = 3$ for at least one value of k are :

(a) $(W_1, W_6), (W_1, W_8), (W_5, W_6), (W_5, W_8)$ for $k = 3$, (b) $(W_2, W_3), (W_2, W_5), (W_3, W_8)$ for $k = 1$, (c) $(W_4, W_5), (W_4, W_7), (W_7, W_8)$ for $k = 2$ are such pairs.

(C) pairs which don't fall under above categories $(W_1, W_3), (W_1, W_5), (W_1, W_7), (W_2, W_4), (W_2, W_6), (W_2, W_8), (W_3, W_5), (W_3, W_7), (W_4, W_6), (W_4, W_8), (W_5, W_7), (W_6, W_8)$ are such pairs.

We consider the above three types of pairs separately.

A-Type pairs:

The k^{th} qubits of the pair for which $\dim(E_{ij}^{mn}) = 2$ is put in lab-B, whereas the i^{th} and the j^{th} together in lab-A. Under this arrangement, any given pair of this type, for a proper choice of basis, reduces to the form:

$$|W_m\rangle_{123} = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B$$

$$|W_n\rangle_{123} = \sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|0\rangle_B$$

(The subscripts A and B indicates the laboratories occupying the qubits.)

Thus as far as LOCC between the labs is concerned, the two states of a given pair of this type are equivalent to two equally entangled states of two qubits lying in different planes. Impossibility of cloning of such states by LOCC with a known state having same entanglement has extensively been discussed in [10]. We simply conclude that W-states falling under the said category cannot be cloned by LOCC between the two labs and hence by the usual LOCC.

B-Type pairs:

Once again the k^{th} qubits of the pair for which $\dim(E_{ij}^{mn}) = 3$ is put in lab-B, whereas the i^{th} and the j^{th} together in lab-A. For ij vs. k cut and for a proper choice of basis [14], a given pair of this type can be written either as :

(I)

$$(W_m)_I = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B$$

$$(W_n)_I = \sqrt{\frac{2}{3}}|1\rangle_A|0\rangle_B + \sqrt{\frac{1}{3}}|2\rangle_A|1\rangle_B$$

or as:

(II)

$$(W_m)_{II} = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B$$

$$(W_n)_{II} = \sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_B + \sqrt{\frac{2}{3}}|2\rangle_A|0\rangle_B$$

Assume now existence of a cloner which, by LOCC between the labs, can clone a pair $(W_m)_I$ and $(W_n)_I$ when a known W-state (suppose W_1) is supplied to it as blank copy. If we supply to the cloner an equal mixture of $(W_m)_I$ and $(W_n)_I$ together with the blank copy W_1 , i.e. if the input state to this LOCC-cloner is:

$$\rho_{in} = \frac{1}{2}P[(W_m)_I \otimes (W_1)] + \frac{1}{2}P[(W_n)_I \otimes W_1],$$

the output of the cloner will be

$$\rho_{out} = \frac{1}{2}P[(W_m)_I \otimes (W_m)_I] + \frac{1}{2}P[(W_n)_I \otimes (W_n)_I]$$

Here P stands for projector.

For proving impossibility of LOCC-cloning of these states, we make use of the fact that Negativity, of a bipartite quantum state ρ , $N(\rho)$ cannot increase under LOCC [12]. $N(\rho)$ is given by [13]

$$N(\rho) \equiv \|\rho^{T_B}\| - 1 \quad (4)$$

where ρ^{T_B} is the partial transpose with respect to system B and $\|\dots\|$ denotes the trace norm which is defined as,

$$\|\rho^{T_B}\| = \text{tr}(\sqrt{\rho^{T_B\dagger}\rho^{T_B}}) \quad (5)$$

Numerical calculations for negativities gives:

$$N(\rho^{in}) = 1.89097; N(\rho^{out}) = 2.14597.$$

As negativity cannot be increased under LOCC between the two labs, hence these W states cannot be cloned.

Negativity calculations for type (II) pairs gives: $N(\rho^{in}) = 2.23802; N(\rho^{out}) = 2.49298.$

where

$$\rho_{in} = \frac{1}{2}P[(W_m)_{II} \otimes (W_1)] + \frac{1}{2}P[(W_n)_{II} \otimes W_1]$$

$$\rho_{out} = \frac{1}{2}P[(W_m)_{II} \otimes (W_m)_{II}] + \frac{1}{2}P[(W_n)_{II} \otimes (W_n)_{II}]$$

and P as usual stands for the projector. As $N(\rho^{in}) < N(\rho^{out})$, hence states belonging to this pair too cannot be cloned.

C-Type pairs:

Every pair of this set has an important feature that there is one value of k for which $\dim(E_{ij}^{mn}) = 4$ and $[\rho_{ij}^m, \rho_{ij}^n] \neq 0$. For showing impossibility of cloning, we put those two qubits together in lab-A for which the reduced density matrices of the corresponding W-pairs are noncommuting. The states of a given pair under this arrangement reduce to the following representative form:

$$|W_m\rangle = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B$$

and

$$|W_n\rangle = \sqrt{\frac{2}{3}}|0'\rangle_A|0\rangle_B + \sqrt{\frac{1}{3}}|1'\rangle_A|1\rangle_B$$

for proper choice of basis [15], where:

$$\langle 0|1\rangle_A = \langle 0'|1'\rangle_A = 0, \langle 0|1'\rangle_A = \langle 0'|1\rangle_A = 0, \\ \langle 0|0'\rangle_A = -\langle 1|1'\rangle_A \text{ and } |\langle 0|0'\rangle_A| = \frac{1}{\sqrt{2}}.$$

Analysis similar to the previous one shows that negativities of the input (equal mixture of W_m and W_n together with a known W state) and output of the assumed cloner (equal mixture of W_m and W_n) are $N(\rho^{in}) = 2.23802$ and $N(\rho^{out}) = 2.55185$ respectively; again denying the existence of such a cloner. \square

One of the outstanding feature of quantum mechanics is that non-orthogonal states can not be cloned. But cloning of orthogonal entangled states using LOCC with appropriate supply of entanglement is another area which would further reveal nature of (multipartite) entanglement and as well as of LOCC. The result of this letter established one stark difference between two kind of symmetric three partite genuine entanglement, namely W-type and GHZ-type entanglement even for a pair of entangled state (where LOCC distinguishability is blunt).

ACKNOWLEDGMENTS

R.R acknowledges the support by CSIR, Government of India, New Delhi.

* Electronic address: sujitr@isical.ac.in

† Electronic address: gkar@isical.ac.in

‡ Electronic address: skunkri@isical.ac.in

§ Electronic address: ramijr@isical.ac.in

¶ Electronic address: anirb@qis.ucalgary.ca

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- [14] (i) Take for example the pair (W_1, W_6) . These states in the '12 vs. 3' cut reduce to form (I) for the substitution: $|0\rangle_A$ for $-|10\rangle_{12}$, $|1\rangle_A$ for $(\frac{100+11}{\sqrt{2}})_{12}$, $|2\rangle_A$ for $|01\rangle_{12}$, $|0\rangle_B$ for $-|0\rangle_3$ and $|1\rangle_B$ for $|1\rangle_3$.
(ii) Take another pair (W_1, W_8) . This for the '12 vs. 3' cut can be written as (II) under the substitution: $|0\rangle_A$ for $|10\rangle_{12}$, $|1\rangle_A$ for $(\frac{100+11}{\sqrt{2}})_{12}$, $|2\rangle_A$ for $(\frac{100-11}{\sqrt{2}})_{12}$, $|0\rangle_B$ for $|0\rangle_3$ and $|1\rangle_B$ for $|1\rangle_3$.
- [15] For example the reduced density matrices of W_1 and W_3 are noncommuting when the third qubits are traced out. So we keep the first and the second qubits in lab-A and the third in lab-B for LOCC between these labs. Under the substitution: $|0\rangle_A$ for $|10\rangle_{12}$, $|1\rangle_A$ for $(\frac{100+11}{\sqrt{2}})_{12}$, $|0'\rangle_A$ for $(\frac{101-10}{\sqrt{2}})_{12}$, $|1'\rangle_A$ for $|00\rangle_{12}$, $|0\rangle_B$ for $|0\rangle_3$ and $|1\rangle_B$ for $|1\rangle_3$; W_1 and W_3 reduce to the said form.